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ANISOTROPIC DISC LOADED BY A LAYER OF FORCES IN AN UNBOUNDED ELASTIC MEDIUM

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We consider an anisotropic ellipsoidal disc in an unbounded isotropic elastic medium under the influence of a uniform stress field $\sigma_0^{\alpha\beta}$. In addition, the disc surface is loaded by a layer of bulk forces $q^\alpha = p^{\alpha\beta} n_\beta$ ($p^{\alpha\beta}$ is a constant symmetric tensor, n_β are the components of a unit normal vector $\mathbf{n}$ to the disc surface). By a disc is meant an ellipsoid one of whose axes is much smaller than the other two.

It has been shown in [1] that the stresses $\sigma^{\alpha\beta}(\mathbf{n})$ on the surface of an anisotropic ellipsoidal inclusion are of the form

$$\begin{gathered}\sigma(\mathbf{n}) = B(\mathbf{n})\varepsilon^+, \quad \varepsilon^+ = B^{-1}(\sigma_0 + c_0\, A\, p), \\ B(\mathbf{n}) = c_0 + c_0\, K(\mathbf{n})(c - c_0),\end{gathered} \tag{1}$$

where ε^+ are the strains inside the inclusion; c_0 and c are the tensors of elastic constants of the ambient medium and the inclusion, respectively; $K(\mathbf{n})$ is the Fourier image of the second derivative of the Green's tensor of the external homogeneous medium; B^{-1} is the tensor inverse to B; $A = \langle K(\mathbf{n})\rangle$ and $B = \langle B(\mathbf{n})\rangle$ are the average values of the tensors $K(\mathbf{n})$ and $B(\mathbf{n})$ over the ellipsoid.

First we solve the problem on strain distribution inside the disc. Let us write the formula for ε^+:

$$\varepsilon^+ = B^{-1}\sigma_0 + B^{-1}c_0\, A\, p = B^{-1}\sigma_0 + R\, p = \varepsilon_1^+ + \varepsilon_2^+. \tag{2}$$

Here ε_1^+ is the strain due to the action of the external field σ_0; ε_2^+ is the strain due to the load distributed over the surface. It is evident from (2) that calculation of the strains inside the disc is reduced to that of the tensors B^{-1} and R.

Let the half-axes of the ellipsoid a_1, a_2, and a_3 satisfy the relationship for $a_3 \ll a_2 \leqslant a_1$. The case $a_2 \ll a_1$ corresponds to a prolate disc; $a_2 \approx a_1$, to a disc approximating a circle; and $a_1 = a_2$, to a circular disc. Let us introduce a small (but finite) parameter $\xi = a_3/a_2$. Then, for a constant tensor A, the following expansion holds:

$$A = A_0 + O(\xi), \quad A_0 = \frac{\alpha}{2\pi}\int\limits_0^{2\pi} \frac{K(\varphi)\, d\varphi}{\cos^2\varphi + \alpha^2 \sin^2\varphi}, \quad \alpha = a_2/a_1. \tag{3}$$

One can show that for an arbitrary anisotropic medium $K(\varphi)$ is a constant tensor, with

$$K(\varphi) = K(\mathbf{n})\Big|_{n_3=1,\ n_1=n_2=0}.$$

Taking into account that

$$\frac{\alpha}{2\pi}\int\limits_0^{2\pi} \frac{d\varphi}{\cos^2\varphi + \alpha^2\sin^2\varphi} = 1,$$

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we find

$$A_0 = K(\mathbf{n})\Big|_{n_3=1,\ n_1=n_2=0}.$$

Consequently, A_0 is a constant tensor independent of the inclusion geometry.

For an isotropic external medium, from the explicit formulas for $K(\mathbf{n})$ [2] it follows that with $n_3 = 1$ only the components $K_{\alpha 3\alpha 3}(n)$ $(\alpha = 1,2,3)$ are nonzero. For the components of the tensor A_0 we find

$$A^0_{3333} = \frac{1-2\nu_0}{2\mu_0(1-\nu_0)}, \quad A^0_{1313} = A^0_{2323} = \frac{1}{4\mu_0}, \tag{4}$$

where μ_0 and ν_0 are the shear modulus and Poisson's coefficient of an isotropic external medium.

Applying expansion (3) for A, we obtain the appropriate expansion with respect to the small parameter ξ of the constant tensor B:

$$B = B_0 + O(\xi).$$

The leading term of the expansion B_0 is the constant tetravalent tensor nonsymmetric with respect to permutation of pairs of indices

$$B_0 = c_0\, A_0\, (c - c_0).$$

Hence and from the properties of the tensor A_0 it follows that for an arbitrary anisotropic medium the tensor B_0 has the same value for prolate, circular, and close to circular discs. One can show that B_0 is a nondegenerate tensor. Therefore, to calculate the tensors B^{-1} and R let us restrict ourselves to the leading terms of the expansion with respect to the small parameter ξ.

Assume the disc to be orthotropic and the axes of elastic symmetry to be parallel to the ellipsoid axes. Then the tensor $c^{\alpha\beta\lambda\mu}$ has nine nonzero components which will be denoted as

$$c^{\alpha\alpha\beta\beta} = c_{\alpha\beta} \quad (\alpha,\ \beta = 1,2,3),$$
$$c^{2323} = c_{44}, \quad c^{1313} = c_{55}, \quad c^{1212} = c_{66}.$$

For the components of the tensor B_0^{-1}, we obtain

$$\begin{gathered} B^{-1}_{1111} = B^{-1}_{2222} = \frac{1}{E_0}, \ B^{-1}_{1122} = B^{-1}_{2211} = B^{-1}_{1133} = B^{-1}_{2233} = -\frac{\nu_0}{E_0}, \\ B^{-1}_{3333} = \frac{1}{c_{33}}\left[1 + \frac{\nu_0(c_{13}+c_{23})}{E_0}\right], \ B^{-1}_{1313} = \frac{1}{4c_{55}}, \ B^{-1}_{2323} = \frac{1}{4c_{44}}, \\ B^{-1}_{3311} = \frac{\nu_0 c_{23} - c_{13}}{c_{33}\, E_0}, \quad B^{-1}_{3322} = \frac{\nu_0 c_{13} - c_{23}}{c_{33}\, E_0}, \quad B^{-1}_{1212} = \frac{1}{4\mu_0} \end{gathered} \tag{5}$$

(E_0 is Young's modulus of the isotropic external medium).

The strain components ε_1^+ are obtained by contracting the tensors B^{-1} and σ_0. Let us define the strains ε_2^+. The leading term of the expansion of the tensor R from (2) is

$$R_0 = B_0^{-1}\, c_0\, A_0.$$

It follows from (4) that the components $R^0_{\alpha\beta 11} = R^0_{\alpha\beta 22} = R^0_{\alpha\beta 12} = 0$ $(\alpha,\beta = 1,2,3)$. This implies that the forces applied to the surface p^{11}, p^{22}, p^{12}, which are parallel to the plane of the disc edge $(n_3 = 0)$, do not contribute to the leading terms of the expansion ε_2^+. The remaining components of the tensor R_0 are

$$R^0_{1133} = R^0_{2233} = 0, \quad R^0_{3333} = \frac{1}{c_{33}}, \quad R^0_{1313} = \frac{1}{4\,c_{55}}, \quad R^0_{2323} = \frac{1}{4\,c_{44}}.$$

Hence it follows that the strains ε_2^+ have the form

$$(\varepsilon_2^+)_{33} = \frac{p^{33}}{c_{33}}, \quad (\varepsilon_2^+)_{13} = \frac{p^{13}}{2\,c_{55}}, \tag{6}$$

$$(\varepsilon_2^+)_{23} = \frac{p^{23}}{2\,c_{44}}, \quad (\varepsilon_2^+)_{\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2).$$

Thus, only the forces p^{33}, p^{13}, and p^{23} (normal to the edge plane of the disc) contribute to the leading terms for ε_2^+.

From (2), (5), and (6) we determine the summary strain ε^+:

$$\varepsilon_{\alpha\beta}^+ = \varepsilon_{\alpha\beta}^0 \quad (\alpha, \beta = 1, 2), \quad \varepsilon_{13}^+ = \frac{\sigma_0^{13} + p^{13}}{2\,c_{55}}, \quad \varepsilon_{23}^+ = \frac{\sigma_0^{23} + p^{23}}{2\,c_{44}},$$

$$\varepsilon_{33}^+ = \frac{1}{c_{33}}[\sigma_0^{33} + p^{33} - c_{13}\varepsilon_{11}^0 - c_{23}\varepsilon_{22}^0].$$

Here $\varepsilon_{\alpha\beta}^0$ are the external strains connected with the stresses $\sigma_0^{\lambda\mu}$ by Hooke's law $\sigma_0^{\lambda\mu} = c_0^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^0$.

From the formulas for ε^+ it is evident that the strains inside the orthotropic disc depend only on the elastic constants that characterize the elastic properties of the disc in the direction orthogonal to the edge plane. Let us write the stresses $\sigma_+^{\alpha\beta}$ inside the orthotropic disc as follows:

$$\sigma_+^{11} = \frac{1}{c_{33}}\,[\Delta_{22}\,\varepsilon_{11}^0 - \Delta_{12}\,\varepsilon_{22}^0 + c_{13}\,(\sigma_0^{33} + p^{33})],$$

$$\sigma_+^{22} = \frac{1}{c_{33}}\,[\Delta_{11}\,\varepsilon_{22}^0 - \Delta_{12}\,\varepsilon_{11}^0 + c_{23}\,(\sigma_0^{33} + p^{33})],$$

$$\sigma_+^{12} = \frac{c_{66}}{\mu_0}\,\sigma_0^{12}, \quad \sigma_+^{\alpha 3} = \sigma_0^{\alpha 3} + p^{\alpha 3} \quad (\alpha = 1, 2, 3),$$

where $\Delta_{\alpha\beta}$ is the cofactor of the element $c_{\alpha\beta}$ in the matrix of elastic constants $\|c_{\alpha\beta}\|$ $(\alpha, \beta = 1, 2, 3)$.

One can show that the stresses σ_+ inside the orthotropic disc depend on seven elastic constants and do not depend on the shear modules c_{44} and c_{55} in the planes orthogonal to the edge plane of the disc.

For the particular case of an isotropic disc with shear modulus μ and Poisson's coefficient ν we have

$$\sigma_+^{11} = \frac{1}{1-\nu}\,[\,2\,\mu\,(\varepsilon_{11}^0 + \nu\,\varepsilon_{22}^0) + \nu\,(\sigma_0^{33} + p^{33})\,],$$

$$\sigma_+^{22} = \frac{1}{1-\nu}\,[\,2\,\mu\,(\varepsilon_{22}^0 + \nu\,\varepsilon_{11}^0) + \nu\,(\sigma_0^{33} + p^{33})\,],$$

$$\sigma_+^{12} = \frac{\mu}{\mu_0}\sigma_0^{12}, \quad \sigma_+^{\alpha 3} = \sigma_0^{\alpha 3} + p^{\alpha 3} \quad (\alpha = 1, 2, 3).$$

Before calculating the stresses $\sigma(\mathbf{n})$ at the outer surface of the disc from (1), we find the values $\sigma^{\alpha\beta}$ at its vertices $A(a_1, 0, 0)$, $B(0, a_2, 0)$, $C(0, 0, a_3)$. For an orthotropic disc at the vertex A

$$\sigma^{\alpha 1}(A) = \sigma_+^{\alpha 1} \quad (\alpha = 1, 2, 3), \quad \sigma^{23}(A) = \frac{\mu_0}{c_{44}}\,(\sigma_0^{23} + p^{23}),$$

$$\sigma^{22}(A) = \frac{1}{1-\nu_0}\,[\,2\,\mu_0\,(\varepsilon_{22}^0 + \nu_0\,\varepsilon_{33}^+) + \nu_0\,\sigma_+^{11}],$$

$$\sigma^{33}(A) = \frac{1}{1-\nu_0}\,[\,2\,\mu_0\,(\varepsilon_{33}^+ + \nu_0\,\varepsilon_{22}^0) + \nu_0\,\sigma_+^{11}\,].$$

Expressions for $\sigma^{\alpha\beta}(B)$ are derived by changing the indices $1 \leftrightarrow 2$, $4 \rightarrow 5$ in both sides of the formulas. At the vertex C

$$\sigma^{\alpha 3}(C) = \sigma_0^{\alpha 3} + p^{\alpha 3} \quad (\alpha = 1, 2, 3), \quad \sigma^{12}(C) = \sigma_+^{12},$$

$$\sigma^{11}(C) = \frac{1}{1-\nu_0}\,[\,2\,\mu_0\,(\varepsilon_{11}^0 + \nu_0\varepsilon_{22}^0) + \nu_0\,(\sigma_0^{33} + p^{33})] = \sigma_0^{11} + \frac{\nu_0}{1-\nu_0}\,p^{33},$$

$$\sigma^{22}(C) = \frac{1}{1-\nu_0}[\,2\,\mu_0\,(\varepsilon_{22}^0 + \nu_0\,\varepsilon_{11}^0) + \nu_0\,(\sigma_0^{33} + p^{33})] = \sigma_0^{22} + \frac{\nu_0}{1-\nu_0}p^{33}.$$

For an isotropic disc the structure of the formulas for stresses at the disc vertices is the same as for an orthotropic disc.

It is more convenient to study the stresses at the disc surface using the local coordinate system $e_{\alpha'}$ related to the normal to the surface so that the axis $e_{3'}$ is directed to the normal $\mathbf{n}$, while the axes $e_{1'}$ and $e_{2'}$ are in the plane tangential to the surface, with the local stress $\sigma^{1'1'}(\mathbf{n})$ being directed perpendicular to the section plane in the "vertical" sections $n_1 = 0$ and $n_2 = 0$, and $\sigma^{2'2'}(\mathbf{n})$, along the section contour, and in the section $n_3 = 0$, vice versa. We denote local stresses by $\sigma^{\alpha\beta}(\mathbf{n})$ (in distinction to $\sigma^{\alpha\beta}(\mathbf{n})$ in a rigid coordinate system). Since the general expressions for $\sigma_{\alpha\beta}(\mathbf{n})$ are cumbersome, we present them only for the main sections of the disc. In the section $n_1 = 0$ $(n_2^2 + n_3^2 = 1)$

$$\begin{gathered}
\sigma_{11}(\mathbf{n}) = n_3^2\,\sigma^{11}(C) + n_2^2\sigma^{11}(B) + 2\,\nu_0\, n_2\, n_3\, (\sigma_+^{23} - 2\,\mu_0\,\varepsilon_{23}^+),\\
\sigma_{22}(\mathbf{n}) = n_3^2\sigma^{22}(C) + n_2^2\,\sigma^{33}(B) + 2\,n_2\,n_3\,(\nu_0\,\sigma_+^{23} - 2\,\mu_0\,\varepsilon_{23}^+),\\
\sigma_{33}(\mathbf{n}) = n_2^2\sigma_+^{22} + n_3^2\,(\sigma_0^{33} + p^{33}) + 2\,n_2\,n_3\,(\sigma_0^{23} + p^{23}),\\
\sigma_{12}(\mathbf{n}) = n_3\,\sigma_0^{12} - 2\,n_2\,\mu_0\,\varepsilon_{13}^+,\\
\sigma_{13}(\mathbf{n}) = n_3\,(\sigma_0^{13} + p^{13}) + n_2\,\sigma_+^{12},\\
\sigma_{23}(\mathbf{n}) = n_2\,n_3\,[\sigma_+^{22} - (\sigma_0^{33} + p^{33})] + (n_3^2 - n_2^2)(\sigma_0^{23} + p^{23}).
\end{gathered} \tag{7}$$

Expressions for $\sigma_{\alpha\beta}(\mathbf{n})$ in the section $n_2 = 0$ are obtained by changing $1 \leftrightarrow 2$, $4 \to 5$, $B \to A$ on the right-hand sides of the above formulas, and in section $n_3 = 0$ they have the form

$$\begin{gathered}
\sigma_{11}(\mathbf{n}) = n_1^2\,\sigma^{22}(A) + n_2^2\sigma^{11}(B) + 2\,n_1\,n_2\,(\nu_0\,\sigma_+^{12} - 2\,\mu_0\,\varepsilon_{12}^+),\\
\sigma_{22}(\mathbf{n}) = n_1^2\,\sigma^{33}(A) + n_2^2\,\sigma^{33}(B) + 2\,\nu_0\,n_1\,n_2\,(\sigma_+^{12} - 2\,\mu_0\,\varepsilon_{12}^+),\\
\sigma_{33}(\mathbf{n}) = n_1^2\,\sigma_+^{11} + n_2^2\,\sigma_+^{22} + 2\,n_1 n_2\,\sigma_+^{12},\\
\sigma_{12}(\mathbf{n}) = 2\,\mu_0\,(n_1\,\varepsilon_{23}^+ - n_2\,\varepsilon_{13}^+),\\
\sigma_{13}(\mathbf{n}) = n_1\,n_2\,(\sigma_+^{11} - \sigma_+^{22}) + (n_2^2 - n_1^2)\,\sigma_+^{12},\\
\sigma_{23}(\mathbf{n}) = n_1\,(\sigma_0^{13} + p^{13}) + n_2\,(\sigma_0^{23} + p^{23}).
\end{gathered} \tag{8}$$

From the formulas derived it is obvious that the stresses at the disc surface calculated up to the leading term of the expansion with respect to the small geometric parameter depend on all components of the external field σ_0 and on the forces $p^{\alpha 3}$ $(\alpha = 1, 2, 3)$ applied to the plane and normal to the plane of the disc edge only. The qualitative picture of the behavior of the stresses affected by the external field σ_0 and the total field $(\sigma_0 + p)$ only is the same: under the effect of tensile stresses $\sigma_0^{\alpha\alpha} + p^{\alpha\alpha}$ $(\alpha = 1, 2, 3)$ the maximum of the stresses $\sigma_{\alpha\alpha}(\mathbf{n})$ is achieved at the disc vertices; under the action of shear stresses $\sigma_0^{\alpha\beta} + p^{\alpha\beta}$ $(\alpha \neq \beta)$ it is achieved in the vicinity of the vertices.

The shear components of the stresses $\sigma_{\alpha\beta}(\mathbf{n})$ $(\alpha \neq \beta)$ can also undergo displacement of the maximum from the vertices under the action of tensile stresses only.

It should be noted that for the case of an isotropic external medium the structure of formulas (7) and (8) for the stresses on the disc surface does not depend on the anisotropy of the inclusion: the expressions for stresses in orthotropic and isotropic discs are identical to the appropriate ε^+ and σ^+. Anisotropy of the external medium introduces significant changes in the formulas for stresses.

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